CONTRIBUTION TO THE THEORY OF STABILITY OF A PLANE LAYER OF GRAVITATING FLUID WITH DENSITY DECAYING EXPONENTIALLY WITHIN A MAGNETIC FIELD

(K TEORII USTOICHIVOSTI PLOSKOGO SLOIA GRAVITIRUIUSHCHEI ZHIDKOSTI S EKSPONENTSIAL'NO UBYVAIUSHCHEI PLOTNOST'IU PRI NALICHII MAGNITNOGO POLIA)

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A study is made of the stability of a plane layer of gravitating fluid with exponentially decaying density within a magnetic field with respect to surface disturbances of periodic character.

A layer of gravitating fluid of thickness $2h$ and density

$$\rho = \rho_0 \exp\{-\beta y\} \qquad (\beta > 0) \tag{1}$$

is imagined to be in equilibrium in a field of natural gravitational force under the influence of an internal uniform magnetic field directed along the x-axis. It is assumed that the xz plane coincides with the axis of symmetry of the layer and the y-axis is directed vertically upwards. The y in Expression (1) should be taken as the absolute value of y.

The equation of the disturbed layer surface, in the chosen coordinate system, is of the form

$$y = h + \delta y = h + a \cos kx \tag{2}$$

Stability investigations of this system in terms of disturbances of type (2) are carried out on an energy basis using the following system of equations:

$$\nabla^2 U = 0, \qquad \nabla^2 V = 4\pi G\rho \tag{3}$$

$$\partial H / \partial t = \frac{1}{4\pi\sigma} \nabla^2 \mathbf{H} + \operatorname{rot}(\mathbf{v} \times \mathbf{H}), \qquad \nabla \mathbf{H} = 0 \tag{4}$$

$$\partial\rho / \partial t + (\mathbf{v}\cdot\nabla)\rho = 0, \qquad \nabla \mathbf{v} = 0 \tag{5}$$

In these expressions V and U are potentials within and outside the medium respectively, G is the gravitational constant, v the velocity, σ is the electrical conductivity of the medium; Equations (5) are derived from the equation of continuity in terms of a proposed model or representation of a gravitating fluid [1]. The boundary conditions for equations (3) to (5) are

$$\left(\frac{\partial V}{\partial y}\right)_{y=0} = 0, \qquad (V)_{y=h} - (U)_{y=h} = 0, \qquad \left(\frac{\partial V}{\partial y}\right)_{y=h} - \left(\frac{\partial U}{\partial y}\right)_{y=h} = 0 \quad (6)$$

$$(v_y)_{y=0} = 0, \qquad (v_y)_{y=h} = \frac{da}{dt} \cos kx \quad (7)$$

In the undisturbed state $\delta y = 0$, $v = 0$ and Equations (4) and (5) are satisfied, whilst the solutions to (3), taking into account (1) and (6) take the following form

$$V_0 = \frac{4\pi G \rho_0}{\beta^2} (e^{-\beta y} + \beta y - 1), \qquad U_0 = \frac{4\pi G \rho_0}{\beta} (1 - e^{-\beta h})\, y + C_0 \quad (8)$$

In this case

$$C_0 = \frac{4\pi G \rho_0}{\beta^2} e^{-\beta h} (1 + \beta h - e^{\beta h}) \quad (9)$$

In the limiting case when $\beta \to 0$ a potential expression is obtained for a plane layer (of constant density ρ_0) from (8) [2,3].

The basic system of Equations (3) to (5) has a solution for the equilibrium condition so that a solution describing a condition of small disturbances can be represented thus

$$\rho = \rho_0 e^{-\beta y} + \delta\rho_\beta, \qquad V = V_0 + \delta V, \qquad U = U_0 + \delta U_0, \qquad \mathbf{H} = \mathbf{H}_0 + \delta\mathbf{H}$$

$$\mathbf{v} = -\operatorname{grad} \varphi(x, y) \quad (10)$$

Here $\delta\rho_\beta$ is the change in density within the medium caused by mass redistribution due to the deformation of the free surface of the layer; the appearance of such a change is caused by the undisturbed layer density gradient so that $\delta\rho_\beta \to 0$ when $\beta \to 0$ and the function $\phi(x, y)$ satisfies the Laplace equation.

A solution of $\phi(x, y)$ satisfying conditions (7) is of the form

$$\varphi(x, y) = -\frac{1}{k} \frac{da}{dt} \frac{\cosh ky}{\sinh kh} \cos kx \quad (11)$$

The velocity distribution consistent with this is

$$v_x = -\frac{da}{dt} \frac{\cosh ky}{\sinh kh} \sin kx, \qquad v_y = \frac{da}{dt} \frac{\sinh ky}{\sinh kh} \cos kx \quad (12)$$

If we neglect products of the type $v_x \partial \delta \rho_\beta / \partial x$ and $v_y \partial \delta \rho_\beta / \partial y$, using (12), we can reduce (5) to the following form

$$\frac{\partial}{\partial t} \delta\rho_\beta = \frac{da}{dt} \rho_0 \beta e^{-\beta y} \frac{\sinh ky}{\sinh kh} \cos kx$$

Remembering that $a = 0$ when $t = 0$ we find

$$\delta\rho_\beta = a\beta\rho_0 e^{-\beta y} \frac{\sinh ky}{\sinh kh} \cos kx \tag{13}$$

The quantity $\delta\rho_\beta$ takes on positive or negative values depending on $\cos kx$. Within the framework of linear theory, therefore, the condition $\delta\rho_\beta \ll \rho$ should be fulfilled, which is transformed into $a\beta \ll 1$ after substituting into the basic inequality of the corresponding expressions from (1) and (13).

The change in potential within and outside the medium will satisfy the following equations respectively

$$\frac{\partial^2 \delta V}{\partial x^2} + \frac{\partial^2 \delta V}{\partial y^2} = 4\pi G \beta \rho_0 a e^{-\beta y} \frac{\sinh ky}{\sinh kh} \cos kx, \qquad \frac{\partial^2 \delta U}{\partial x^2} - \frac{\partial^2 \delta U}{\partial y^2} = 0 \tag{14}$$

Note that

$$\delta\rho_\beta = \begin{cases} a\beta\rho_0 e^{-\beta y} \frac{\sinh ky}{\sinh kh} \cos kx & \text{when} -Y \leqslant y \leqslant Y \\ 0 & \text{when} \quad Y > y > Y \end{cases}$$

$$\int_{-\infty}^{+\infty} |\delta\rho_\beta| \, dy = \int_{-Y}^{+Y} |\delta\rho_\beta| \, dy < \infty \qquad (Y = h + a\cos kx)$$

Therefore the particular integral of (14) can be represented as a Fourier integral. Put

$$\begin{aligned} \delta\rho_\beta &= \left(\frac{1}{\pi} \int_0^\infty d\alpha \int_{-\infty}^{+\infty} \delta\rho_\beta(\tau) \cos\alpha(\tau - y) \, d\tau\right) \cos kx \\ \delta V &= \left(\frac{1}{\pi} \int_0^\infty d\alpha \int_{-\infty}^{+\infty} \delta V(\tau) \cos\alpha(\tau - y) \, d\tau\right) \cos kx \end{aligned} \tag{15}$$

Thus on substituting (15) in (14) we arrive at the following relationship between the Fourier components

$$\delta V(\tau) = -\frac{4\pi G}{k^2 + \alpha^2} \delta\rho_\beta(\tau) \tag{16}$$

If follows from (15) and (16)

$$\delta V(x, y) = R \int_0^{\infty} \frac{d\alpha}{k^2+\alpha^2} \int_{-Y}^{+Y} [e^{k_1\tau} - e^{k_2\tau}] \cos(\tau - y)\, d\tau \tag{17}$$

where

$$R = -\frac{2G\rho_0 aB}{\sinh kh} \cos x, \qquad k_1 = k - \beta, \qquad k_2 = -k - \beta$$

In view of the symmetry of the problem with respect to the $y = 0$ plane, the function $\delta\rho_\beta$ may be considered an even function by taking, in Expression (17) the absolute value of the variable τ.

Therefore

$$\delta V(x, y) = 2R \int_0^{\infty} \frac{d\alpha}{k^2+\alpha^2} \int_0^{+Y} [e^{k_1\tau} - e^{k_2\tau}] \cos\alpha(\tau - y)\, d\tau$$

If we integrate with respect to τ to an accuracy first order in amplitude we arrive at

$$\delta V(x, y) = 2R \left(\int_0^{\infty} \left[\frac{\alpha \cos\alpha y \sin\alpha h\, d\alpha}{(k^2+\alpha^2)(k_1^2+\alpha^2)} + \frac{k_1 \cos\alpha y \cos\alpha h\, d\alpha}{(k^2+\alpha^2)(k_1^2+\alpha^2)} \right] e^{k_1 h} - \right.$$

$$- \int_0^{\infty} \frac{k_1 \cos\alpha y\, dy}{(k^2+\alpha^2)(k_1^2+\alpha^2)} - \int_0^{\infty} \left[\frac{\alpha \cos\alpha y \sin\alpha h\, d\alpha}{(k^2+\alpha^2)(k_2^2+\alpha^2)} + \frac{k_2^2 \cos\alpha y \cos\alpha h\, d\alpha}{(k^2+\alpha^2)(k_2^2+\alpha^2)} \right] e^{k_2 h} +$$

$$\left. + \int_0^{\infty} \frac{k_2 \cos\alpha y\, d\alpha}{(k^2+\alpha^2)(k_2^2-\alpha^2)} \right)$$

The integrals obtained here are easy to transform into Laplace integrals, which, on integrating, lead to a particular integral of (14) in the form

$$\delta V_1 = \frac{2\pi G\rho_0 a}{\sinh kh} \left[\left(1 - \frac{\beta e^{-2kh}}{\beta + 2k} \right) \frac{e^{-\beta h}}{k} \cosh ky - \frac{2\beta e^{-ky}}{\beta^2 - 4k^2} + \frac{e^{(k-\beta)y}}{\beta - 2k} - \frac{e^{-(k+\beta)y}}{\beta + 2k} \right] \cos kx$$

It is evident that the general solution of Equation (14) which satisfies the first boundary condition (6) will be

$$\delta V = \delta V_1 + B \cosh ky \cos kx$$

We will take the solution for δU in this form

$$\delta U = B_1 e^{-k(y-h)} \cos kx$$

By using the second and third boundary conditions in (6) the values of the unknown constants B and B_1 can be calculated.

In the following text the calculation of the change in potential energy will be carried out with the help of the expression for the potential within the medium

$$V = \frac{4\pi G\rho_0}{\beta^2}(e^{-\beta y} + \beta y - 1) - \frac{4\pi G a\rho_0 e^{-\beta h}\cosh kky}{k(1+\tanh kh)\cosh kh}\cos kx +$$

$$+ \frac{2\pi G a\rho_0}{\sinh kh}\left[\frac{e^{-\beta h}}{k}\left(1 - \frac{\beta e^{-2kh}}{\beta + 2k}\right)\cosh ky - \frac{2\beta e^{-ky}}{\beta^2 - 4k^2} + \frac{e^{(k-\beta)y}}{\beta - 2k} - \frac{e^{-(k+\beta)y}}{\beta + 2k}\right]\cos kx \quad (18)$$

The change in potential energy per unit length can be calculated from Formula (2)

$$\delta\Omega = \frac{1}{2\lambda}\int_0^{\lambda}[V]_{y=Y}\sigma\,dx \qquad \begin{pmatrix}\sigma = a\rho_0\exp[-\beta Y]\cos kx \\ Y = h + a\cos kx,\ \lambda = 2\pi/k\end{pmatrix}$$

Substitute into this Expression (18) for V and integrate as far as second order terms with respect to amplitude, and we arrive at

$$\delta\Omega = \pi G a^2 h\rho_0{}^2 e^{-2n}\left\{\frac{e^n(2-n)-2}{n} - \frac{1}{z(1+\tanh z)} + \right.$$

$$\left. + \frac{n + 2z - ne^{-2z}}{2z(n+2t)\tanh z} + \frac{2z\cosh + n\sinh z - ne^{n-z}}{(n^2 - 4z^2)\tanh z}\right\} \qquad \begin{pmatrix}z = kh \\ n = \beta h\end{pmatrix} \qquad (19)$$

We proceed by integrating the first equation of (4) (assuming the medium to have infinite electrical conductivity), and, bearing in mind the velocity distribution (12), we obtain for the intensity of the magnetic field

$$h_x = -kH_0 a\frac{\cosh ky}{\sinh kh}\cos kx$$

$$h_y = -kH_0 a\frac{\sinh ky}{\sinh kh}\sin kx$$

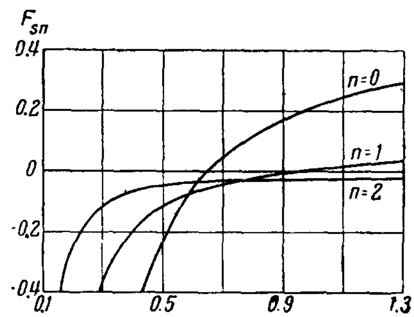

Fig. 1.

Thus the variation in magnetic energy which passes through unit length of disturbed layer will be

$$\delta M = \frac{a^2 H_0^2 z\cosh z}{16\pi h\sinh z} \qquad (20)$$

On adding (19) and (20) we obtain the total change in energy thus

$$\delta E = \delta\Omega + \delta M = \pi G h\rho_0{}^2 a^2 F_{sn}(z) \qquad (21)$$

where

$$F_{sn}(z) = e^{-2n}\left\{\frac{e^n(2-n)-2}{n} - \frac{1}{z(1+\tanh z)} + \frac{n+2z-ne^{-2z}}{2z(n+2z)\tanh z} + \right.$$
$$\left. + \frac{2z\cosh z + n\sinh z - ne^{n-z}}{(n^2-4z^2)\sinh z} + \left(\frac{H_0}{H_s}\right)^2 z\frac{\cosh z}{\sinh z}\right\} \qquad (H_s = 4\pi\rho_0 h\sqrt{G})$$

The total change in energy, therefore, depends on z, n and h_0. Observe that the variation in gravitational energy depends only on n. Figure 1 shows the relation $F_{sn}(z) = \delta E/\pi Gh\rho_0{}^2a^2$ as a function of z, according to (21), for various values of n, when the magnetic field $H_0 = 0$.

In the case where there is no magnetic field, for $n = 0$ ($\rho = \rho_0$) and for $n = 1$ ($\rho = \rho_0 \exp(-y/h)$), function $F_{sn}(z)$, and thus also $\delta\Omega$, take on both positive and negative values. For a definite value $z = z_*$ ($\lambda = \lambda_* = 2\pi h/z_*$) the change in gravitational energy becomes zero ($\delta\Omega = 0$). This value z_* divides the region into stable and unstable harmonics because $\delta\Omega < 0$ when $\lambda < \lambda_*$ and $\delta\Omega > 0$ when $\lambda > \lambda_*$.

When $n \geqslant 2$, $\delta\Omega < 0$ for all values of z or λ. The configuration, therefore, of a type of plane layer whose density varies as $\rho = \rho_0 \exp(-ny/h)$ where $n \geqslant 2$, is extremely unstable and probably does not exist in nature.

Magnetic energy changes are positive for all values of λ; the influence of the magnetic field therefore has a stabilising effect. An increase in intensity of the magnetic field means that the value of z at which the total energy change equals zero ($\delta E = 0$) is reduced, and in this way the region of unstable harmonics is cut down. The most unstable development with $n \geqslant 2$ in the presence of a magnetic field can turn out to be stable within a particular frequency band.

It is apparent therefore that the equilibrium condition of our system as a function of h, H_0, λ may be both stable and unstable. However there may exist a magnetic field which can wholly suppress the development of unstable harmonics. The intensity of such a magnetic field is determined from the condition $\delta E = 0$ and depends on z and n

$$H_0{}^2 \geqslant H_s{}^2 \tanh z e^{-2n}\left\{\frac{1}{z(1+\tanh z)} - \frac{e^n(2-n)-2}{n} + \frac{n+2t-ne^{-2z}}{2z(n+2z)\tanh z} - \right.$$
$$\left. - \frac{2z\cosh z + n\sinh z - ne^{n-z}}{\sinh z\,(n^2-4z^2)}\right\}$$

As a rule the instability is not the same for all values of the length λ which belong to the region of unstable harmonics.

The instability of the system is a maximum for a given value λ_m and the amplitude grows more rapidly. In order to find the maximum unstable harmonic it is essential to obtain an equation of motion with the help of the Lagrange function.

The kinetic energy is

$$\delta T = \frac{1}{2\lambda}\int_0^{\lambda}\int_0^{Y} \rho\,(v_x^2 + v_y^2)\,dx\,dy \qquad (Y = h + \cos kx)$$

If we substitute expressions for v_x and v_y according to (12) into this formula, we obtain on integrating

$$\delta T = \rho_0 h \dot{a}\,\frac{e^{-n}(2z\sinh 2z + n\cosh 2z) - n}{4(4z^2 - n^2)\sinh^2 z} \tag{22}$$

On setting up a Lagrange function using (21) and (22), and integrating, we arrive at

$$\ddot{a} + 4\pi G\rho_0\,\frac{\sinh^2 z\,(4z^2 - n^2)\,F_{sh}(z)}{e^{-n}[2z\sinh 2z + n\cosh 2z] - n}\,a = 0$$

The solution to this equation is

$$a = C\exp\{\pm P_{sn}(z)\,t\}$$

where

$$P_{sn}^2(z) = -4\pi G\rho_0 e^{n}\,\frac{\sinh^2 z\,(4z^2 - h^2)\,F_{sn}(z)}{2z\sinh 2z + n\cosh 2z - ne^{n}} \qquad (c = \mathrm{const})$$

Figures 2 and 3 show graphs of the function $f_{sn} = P_{sn}^2(z)/4\pi G\rho_0$ as a function of z for several definite values of n and H_0/H_s. When $P_{sn}^2(z)/4\pi G\rho_0 < 0$ we have a stable condition because this is conducive to

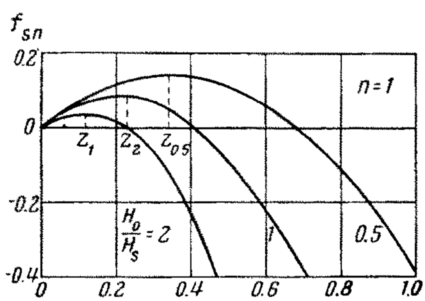

Fig. 2.

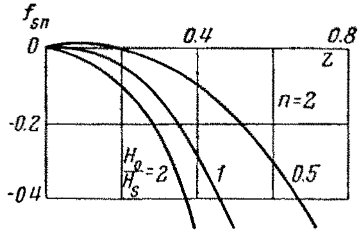

Fig. 3.

periodic variation of amplitude with time. On the other hand the amplitude increases exponentially and the system breaks down into separate parts. The length of these parts along the x-axis is of the order

$\lambda_m = 2\pi h/z_m$ where $z_m(z_{0.5}, z_1, z_2)$ are values of z at which $P_{sn}(z)$ has a maximum and the amplitude increases the most rapidly of all [4,2].

The quantity $\tau^* = P_{sn}^{-1}(z_m)$ is usually understood to be the relaxation time necessary for the unstable condition to develop.

The results of the present paper are more general. For $\beta \to 0$ the results published in [2,3] can be obtained, relating to the stability problems of a constant density plane layer.

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